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Robust Iterative Learning Control Design for Batch Processes with Uncertain Perturbations and Initialization

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A robust iterative learning control (ILC) scheme for batch processes with uncertain perturbations and initial conditions is developed. The proposed ILC design is transformed into a robust control design of a 2-D Fornasini–Marchsini model with uncertain parameter perturbations. The concepts of robust stabilities and convergences along batch and time axes are introduced. The proposed design leads to nature integration of an output feedback control and a feedforward ILC to guarantee the robust convergence along both the time and the cycle directions. This design framework also allows easy enhancement of the feedback and/or feedforward controls of the system by extending the learning information along the time and/or the cycle directions. The proposed analysis and design are formulated as matrix inequality conditions that can be solved by an algorithm based on linear matrix inequality. Application to control injection packing pressure shows the proposed ILC scheme and its design are effective. © 2006 American Institute of Chemical Engineers *AIChE J,* 52: 2171–2187, 2006
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Introduction

Iterative learning control (ILC) is a technique that improves control performance by refining control input from cycle to cycle. The conventional ILC,[1] which learns only from the information of the previous cycles, is an open-loop feedforward control scheme that is ill-suited for the processes with nonrepeatable natures, such as nonrepeatable parameter perturbations and disturbances, and uncertain initial conditions.[2]

It has been proposed to combine iterative learning control with feedback control to improve the robustness and convergence of the ILC system. Based on robust control theory, various frequency methods of iterative learning controller design have been proposed for the continuous-time repetitive processes with model uncertainties.[3-7] In these methods, a feedback controller has been combined with an iterative learning controller for the system robustness and convergence. The impacts of these two types of controllers on the control performances and their relationships, however, have not been investigated. Uncertainties of the initial conditions, which are usually unavoidable in practice, were not addressed in all these works.

The ILC system, where the dynamical behavior along the time is mainly determined by the process and the feedback controller, whereas the feedforward iterative learning introduces the dynamics along the cycle, can be considered as a special two-dimensional (2-D) system. It is advantageous to investigate ILC system design in terms of a 2-D system. First, a 2-D system—a maturing area[8] with many designs and methodologies[9-13]—can be borrowed for ILC. Second, treating the ILC design as a 2-D system allows properties of the ILC system along the time and cycle directions to be separately investigated, which can help to interpret the actions of the feedback controller and the iterative learning controller in the combined system. Because the time duration of a batch process is finite, most existing ILC schemes consider the system robustness and convergence only along the cycle direction. We would argue that taking the control performances along the time direction into account is also very important because it can guarantee the system robustness against nonrepeatable uncertainties, such as nonrepeat-

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able parameter perturbations and disturbances and uncertain initial conditions. Design of ILC as a 2-D system also allows all controllers of the system to be designed in an integral fashion, thus ensuring that all work together harmoniously.

Some ILC designs have indeed been developed based on 2-D system theory.[14-18] However, all of these methods were limited to a linear model without model uncertainties. Galkowski et al.[19] discussed the robust stability analysis and robust controller design for a class of linear repetitive processes that can be described by a 2-D state–space model. Based on a 2-D Roesser system, the authors[20] proposed an integrated design method and investigated the robust convergences of the system along both the time and the cycle directions. However, the above two robust designs require the system state to be measured or estimated, which may not be practical in reality.

This article proposes a new robust ILC design scheme for batch processes with not only uncertain parameter perturbations but also uncertain initial conditions. Robust control for a 2-D Fornasini–Marchsini (FM) model serves as the foundation of the proposed design. For robust convergence investigation of the ILC system along both time and cycle directions, a set of relevant terms of robust stability and convergence from the 2-D system perspective are defined. The resulting stability conditions can be used to analyze the robust convergence of the ILC system not only along the cycle direction but also along the time direction. For easy implementation of the proposed ILC scheme, robust 2-D output feedback control is developed to avoid the state measurement. The proposed design also allows flexible extensions of learning information to be incorporated into the design for enhancement of the system control performance along the time and/or the cycle directions. The analysis and design are formulated as matrix inequality conditions, which can be solved by an algorithm based on linear matrix inequality (LMI). The effectiveness of the proposed ILC scheme and design is illustrated with the applications to the nozzle packing pressure control of injection molding.

The remainder of the article is organized as follows: The mathematical description of the ILC system with uncertain parameter perturbations is presented together with its transformation into a 2-D FM model. Some new stabilities and convergence indices for a 2-D FM system, based on which the robust 2-D output feedback stabilization problems are studied. The system structure of the proposed ILC system and the design algorithm are discussed. Extension of the learning information for ILC enhancement along the time and cycle directions is addressed. The effectiveness of the proposed ILC scheme and design is illustrated. Conclusions are drawn and reported in the final section.

Throughout this article, the following notations are used: $\mathbb{R}^n$ represents Euclidean n space with the norm denoted by $\|\cdot\|$. $\mathbb{R}^{n\times m}$ is a set of $n \times m$ real matrices. For any matrix $M \in \mathbb{R}^{n\times n}$, $M > 0/M \geq 0$ means that M is a positive/semipositive definite symmetric (PDS/SPDS) matrix. M^T represents the transpose of matrix M. I and 0, respectively, denote the identity matrix and the zero matrix with appropriate dimensions. Notation "*" represents the symmetric element of a PDS/SPDS matrix. For a 2-D signal $w(i, j)$, if $\|w\|_{2e} = \sqrt{\sum_{i=0}^{p}\sum_{j=0}^{q}\|w(i,j)\|^2} < \infty$ for any integers $p, q > 0$, then $w(i, j)$ is said to be in ℓ_{2e} space, denoted by $w \in \ell_{2e}$. Notations "$\forall$", "$\Rightarrow$," and "$\Leftrightarrow$" respectively mean "any," "imply," and "equivalent."

ILC System and 2-D Representation

ILC system

A batch process is a process repetitively performing a task over a certain period of time, called *cycle*. In each cycle, the process of interest in this article is assumed to be described by the following discrete-time model with uncertain parameter perturbations:

$$P_\Delta:\begin{cases} x_k(t+1) = (A + \Delta_a(t,k))x_k(t) + (B + \Delta_b(t,k))u_k(t) \\ y_k(t) = Cx_k(t) \end{cases}$$
$$x_k(0) = x_{0k} \qquad 0 \leq t \leq T \qquad k = 1, 2, \cdots \quad (1)$$

where the subscript k indicates the cycle; t is the time index; $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^m$, and $y_k(t) \in \mathbb{R}^l$ are, respectively, the state, input, and output of the process at time t in the kth cycle; x_{0k} represents the initial condition of the kth cycle; and $\{A, B, C\}$ constitute the nominal system matrices with appropriate dimensions. The matrices $\Delta_a(t, k)$ and $\Delta_b(t, k)$, respectively, denote the uncertain parameter perturbations at time t in the kth cycle with the following structures:

$$\Delta_a(t,k) = E_a\Delta_1(t,k)F_a$$
$$\Delta_b(t,k) = E_b\Delta_2(t,k)F_b \quad (2)$$

where $\{E_a, F_a\}$ and $\{E_b, F_b\}$ are known matrices with appropriate dimensions, indicating the structures of the uncertain parameter perturbations, and $\{\Delta_i(t, k)\}_{i=1,2}$ denote the unknown perturbation matrices satisfying the following norm-bounded conditions:

$$\Delta_i^T(t,k)\Delta_i(t,k) < I, \quad i = 1, 2, \quad \forall\, k = 1, 2, \ldots, \quad \forall\, 0 \leq t \leq T \quad (3)$$

Note that $\{\Delta_i(t, k)\}_{i=1,2}$ are generally represented as the functions of both time t and cycle k. If $\{\Delta_i\}_{i=1,2}$ depend on time t only, then the uncertain parameter perturbations are called *repeatable*, otherwise *nonrepeatable*. Although the conventional ILC schemes can effectively deal with repeatable parameter perturbations, practically, it is equally important to consider the control of the batch processes with nonrepeatable parameter perturbations.

For uncertain batch process P_Δ, the iterative learning law of interest has the following general form:

$$\Sigma_{ilc}:\begin{cases} u_k(t) = u_{k-1}(t) + r_k(t) \\ u_0(t) = 0 \end{cases} \quad (4)$$

Here, $r_k(t)$ is referred to as the *updating law* of iterative learning to be determined, $u_0(t)$ represents the *initial values* of iteration. Different updating laws may result in different ILC schemes according to what the available information is and how it is used. For example, for a P-Type ILC, only the tracking errors of the last cycle are proportionally used, leading to the updating law, which has the form $r_k(t) = Le_{k-1}(t)$, where $e_k(t) = y_k(t) - y_r(t)$ represents the tracking error and $y_r(t)$ is the desired trajectory.

For an ILC system, design objectives are stated as follows.

Design Objective 1. For a given uncertain batch process P_Δ and all admissible uncertain parameter perturbations satisfying norm-bounded conditions (Eq. 3) and design iterative learning law (Eq. 4) [or, equivalently, updating law $r_k(t)$] such that

- the control system is robustly stable
- within each cycle, the control system is robustly stable with convergence of any initial tracking error to zero along the time direction as fast as possible
- the control system is also robustly stable with convergence of the tracking error to zero along the iteration/cycle direction as fast as possible
- the possible impacts of the uncertain parameter perturbations to the above control performances should be as small as possible

Equivalent 2-D representation

From the description of ILC system (Eqs. 1–4), it can be clearly seen that all variables of the control system are the functions of both time t and cycle index k; and model P_Δ describes the dynamics of the system along the time, whereas ILC law Σ_{ilc} introduces the dynamics along the cycle and/or the time. In other words, the ILC system is essentially a 2-D system.

Let

$$x(t, k) = x_k(t), \quad u(t, k) = u_k(t), \quad y(t, k) = y_k(t), \quad r(t, k) = r_k(t) \tag{5}$$

According to ILC law Σ_{ilc} and model P_Δ, we have

$$\Delta x(t+1, k) = (A + \Delta_a(t, k))\Delta x(t, k) + (B + \Delta_b(t, k))r(t, k) + w(t, k) \tag{6}$$

where

$$\Delta x(t, k) = x(t, k) - x(t, k-1) \tag{7}$$

$$w(t, k) = (\Delta_a(t, k) - \Delta_a(t, k-1))x(t, k-1) + (\Delta_b(t, k) - \Delta_b(t, k-1))u(t, k-1) \tag{8}$$

Obviously, for any nonrepeatable parameter perturbations, $w(t, k) \neq 0$; otherwise, $w(t, k) = 0$. Thus, we designate $w(t, k)$ as a *nonrepeatable disturbance*.

Now, from the definition $e(t, k) \triangleq y(t, k) - y_r(t)$, it follows that

$$e(t+1, k) = C(A + \Delta_a(t, k))\Delta x(t, k) + e(t+1, k-1) + C(B + \Delta_b(t, k))r(t, k) + Cw(t, k) \tag{9}$$

By combining Eq. 6 with Eq. 9, the following dynamical model is then obtained:

$$\Sigma_p: \begin{cases} \begin{bmatrix} \Delta x(t+1, k) \\ e(t+1, k) \end{bmatrix} = (\tilde{A}_1 + \tilde{\Delta}_a(t, k))\begin{bmatrix} \Delta x(t, k) \\ e(t, k) \end{bmatrix} + \tilde{A}_2\begin{bmatrix} \Delta x(t+1, k-1) \\ e(t+1, k-1) \end{bmatrix} + (\tilde{B} + \tilde{\Delta}_b(t, k))r(t, k) + \tilde{H}w(t, k) \\ y(t, k) \triangleq \begin{bmatrix} e(t, k-1) \\ e(t, k) \end{bmatrix} = \tilde{C}\begin{bmatrix} \Delta x(t, k) \\ e(t, k) \end{bmatrix} \\ z(t, k) \triangleq e(t, k) = \tilde{G}\begin{bmatrix} \Delta x(t, k) \\ e(t, k) \end{bmatrix} \end{cases} \tag{10}$$

where

$$\tilde{A}_1 = \begin{bmatrix} A & 0 \\ CA & 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} I \\ C \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} -C & I \\ 0 & I \end{bmatrix}, \quad \tilde{G} = [0 \quad I]$$

$$\tilde{\Delta}_a = \begin{bmatrix} \Delta_a(t, k) & 0 \\ C\Delta_a(t, k) & 0 \end{bmatrix} = \begin{bmatrix} E_a \\ CE_a \end{bmatrix}\Delta_1(t, k) \times [F_a \quad 0] \triangleq \tilde{E}_a\Delta_1(t, k)\tilde{F}_a$$

$$\tilde{\Delta}_b = \begin{bmatrix} \Delta_b(t, k) \\ C\Delta_b(t, k) \end{bmatrix} = \begin{bmatrix} E_b \\ CE_b \end{bmatrix}\Delta_2(t, k)F_b \triangleq \tilde{E}_b\Delta_2(t, k)\tilde{F}_b \tag{11}$$

Model Σ_p is a typical 2-D Fonasini-Marchesini (FM) model[21] with uncertain parameter perturbations and $[\Delta x^T(t, k)\ e^T(t, k)]^T$, $r(t, k)$, $y(t, k)$, $z(t, k)$, and $w(t, k)$ are viewed, respectively, as the super-state, input, measurement output, controlled output, and disturbance of the 2-D system. Because model Σ_p equivalently represents the dynamical behavior of the tracking error of the ILC system, it is called the *equivalent 2-D tracking error model*. It should be noted from the model that a pair of available information—real-time tracking error $e(t, k)$ and tracking error of the last cycle at current time $e(t, k-1)$—are selected as the controlled outputs to indicate the impact of the nonrepeatable disturbance.

It is shown in model Σ_p that the tracking error is one of the elements of the super-state, which implies that the robust stability and convergence of 2-D model Σ_p at the origin respectively reflect the robust stability and convergence of the ILC system, and the disturbance rejection of model Σ_p to disturbance $w(t, k)$ exactly reflects the rejection performance of the ILC system to the nonrepeatable disturbance. Therefore, it is clear that design of the updating law $r(t, k)$ for the ILC system is equivalent to designing a stabilization control law for the equivalent 2-D tracking error model, which implies that the design and analysis of the ILC system can be solved within the framework of a 2-D system.

Robust Stability and Control of a 2-D FM System

The equivalent 2-D tracking error model Σ_p is a special 2-D FM model with uncertain parameter perturbations. It differs from a general 2-D FM model[9,11,12,21] in that the time duration of each cycle is finite. For simplicity and completeness, in this section, we will discuss the robust stability and control problem from a general 2-D FM model that will be used to analyze and design the ILC system.

Uncertain 2-D FM system

Consider a general 2-D FM system with uncertain parameter perturbations described by the following 2-D dynamical model:

$$\Sigma: \begin{cases} x(t+1, k+1) = (A_1 + \Delta A_1)x(t, k+1) \\ \qquad + (A_2 + \Delta A_2)x(t+1, k) + (B + \Delta B)u(t, k+1) \\ \qquad + Hw(t, k+1) \\ y(t, k) = Cx(t, k) \\ z(t, k) = Gx(t, k) \end{cases}$$

$$t, k = 0, 1, 2, \cdots \tag{12}$$

where $x(t, k) \in \mathbb{R}^n$, $u(t, k) \in \mathbb{R}^m$, $y(t, k) \in \mathbb{R}^l$, $z(t, k) \in \mathbb{R}^q$, and $w(t, k) \in \mathbb{R}^r$ are, respectively, the state, input, measurement output, controlled output, and disturbance of the 2-D system. $\{A_1, A_2, B, C, G, H\}$ represent nominal systems matrices with appropriate dimensions and $\{\Delta A_1, \Delta A_2, \Delta B\}$ are unknown matrices, representing the uncertain parameter perturbations of the system with the following forms:

$$\begin{cases} \Delta A_1 = E_1\Delta_1(t, k)F_1 \\ \Delta A_2 = E_2\Delta_2(t, k)F_2 \\ \Delta B = E_3\Delta_3(t, k)F_3 \end{cases} \tag{13}$$

where $\{E_i, F_i\}_{i=1,2,3}$ are known constant matrices indicating the structures of the uncertainties, and $\{\Delta_i(t, k)\}_{i=1,2,3}$ are unknown matrix functions satisfying the following norm-bounded conditions:

$$\Delta_i^T(t, k)\Delta_i(t, k) < I, \quad i = 1, 2, 3 \tag{14}$$

For 2-D system Σ, the state evolves along two axes, called the *T-axis* and the *K-axis*, respectively. Obviously, the boundary conditions of the 2-D system should be two-dimensional, denoted by

$$\begin{cases} x_{t,0} = x(t, 0), & t = 1, 2, \cdots \\ x_{0,k} = x(0, k), & k = 1, 2, \cdots \end{cases} \tag{15}$$

Here, we designate $x_{t,0}$ as the *K-boundary* and $x_{0,k}$ as the *T-boundary*.

Robust stability

Before the robust stability analysis, the following definitions are necessary.

Definition 3.1.[21] For any bounded boundary conditions $\{x_{t,0}, x_{0,k}\}$ and all admissible uncertain parameter perturbations satisfying conditions of Eq. 14, if the unforced state response of 2-D system Σ satisfies

$$\lim_{t,k\to\infty} x(t, k) = 0 \tag{16}$$

then, the 2-D system Σ is designated *2-D robust asymptotically stable* (for short, *2-D-stable*).

The state of 2-D FM system evolves along two axes and 2-D-stability describes only the robust stability of the system in the 2-D sense. To investigate the robust stabilities for the system along the two axes separately, the following concepts are introduced.

Definition 3.2. Assume K-boundary $x_{t,0}$ is zero. For any T-boundary $x_{0,k}$ and any integer $N > 0$, if the unforced state response of the 2-D system Σ satisfies

$$\lim_{t\to\infty} \sum_{k=1}^{N} \|x(t, k)\| = 0 \tag{17}$$

then, the 2-D system Σ is designated *robust asymptotically T-stable* (for short, *T-stable*).

Definition 3.3. Assume T-boundary $x_{0,k}$ is zero. For any K-boundary $x_{t,0}$ and any integer $N > 0$, if the unforced state response of the 2-D system Σ satisfies

$$\lim_{k\to\infty} \sum_{t=1}^{N} \|x(t, k)\| = 0 \tag{18}$$

then, the 2-D system Σ is designated *robust asymptotically K-stable* (for short, *K-stable*).

Remark 3.1. T-stability and K-stability describe the robust stabilities of the 2-D system along the T-axis and the K-axis. For the equivalent 2-D tracking error model Σ_p, the K-stability implies that the ILC system is robustly stable along the cycle direction and the *sum* of tracking errors with respect to the time index converges from cycle to cycle when there are no initial tracking errors, whereas the T-stability implies that the control system is robustly stable along the time direction and the *sum* of the initial errors with respect to the cycle index converges to zero along the time direction. Most existing iterative learning algorithms[1,3-7] considered only the K-stability of the ILC system, with the result that the control performances within a cycle cannot be guaranteed, especially when there exist nonrepeatable uncertain parameter perturbations and/or the initial condition problem. For many batch processes, uncertainty in initialization is unavoidable.[2] Introducing T-stability and K-stability for a 2-D system not only allows the stabilities along the two axes to be investigated separately but also allows the controller design to be more flexible. These issues are very important for the ILC system. For example, for batch processes with strong nonrepeatable unknown parameter perturbation and/or nonzero initial tracking error, T-stability should be emphasized more; on the other hand, for processes with strong repeatable perturbations, the K-stability should be emphasized more.

To describe the sensitivity of the controlled output to the disturbance, the following definition is extended from robust control theory of the one-dimensional (1-D) system.

Definition 3.4.[22] For a scalar $\gamma > 0$, the 2-D FM system Σ is said to have *robust H_∞ performance* γ if it is 2-D-stable and, for zero boundary conditions and any disturbance $w \in \ell_{2e}$, the unforced response of the system satisfies

$$\|z\|_{2e} < \gamma \|w\|_{2e} \tag{19}$$

Similar to the 1-D system, the robust H_∞ performance γ indicates the maximum sensitivity of the controlled output to the disturbance. A smaller value of γ indicates a better disturbance rejection performance. For controller design, γ should be minimized.

In the 1-D system, Lyapunov's method is widely used for stability and convergence analysis. It can be extended to the 2-D FM system.

Theorem 3.1. An unforced 2-D system Σ is 2-D-stable if there exist a function $V(\cdot)$ and a scalar $\rho > 1$ satisfying

(1) $V(x) \geq 0$ for $\forall x \in \mathbb{R}^n$, and $V(x) = 0 \Leftrightarrow x = 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$

(3) for any boundary conditions

$$\sum_{\substack{t+k=T_0+K_0+i+1 \\ T_0 \leq t \leq T_0+i \\ K_0 \leq k \leq K_0+i}} V(x(t, k)) < \rho^{-1} \sum_{\substack{t+k=T_0+K_0+i \\ T_0 \leq t \leq T_0+i \\ K_0 \leq k \leq K_0+i}} V(x(t, k)),$$

$$\forall\, T_0 > 0,\ K_0 > 0,\ i > 0 \tag{20}$$

Moreover, the maximum value of ρ satisfying Eq. 20 is called the *2-D robust convergence index* (2-D-RCI) of the system.

Proof: For any integers t, $k > 0$, the following inequality results directly from condition 20:

$$V(x(t, k)) < \sum_{i=0}^{t-1} \rho^{-k-i} V(x_{t-i,0}) + \sum_{i=0}^{k-1} \rho^{-t-i} V(x_{0,k-i}) \tag{21}$$

For any boundary conditions $\{x_{t,0}, x_{0,k}\}$, $\rho > 1$ implies

$$\lim_{t,k\to\infty} V(x(t, k)) = 0 \tag{22}$$

From condition (2), it then follows that

$$\lim_{t,k\to\infty} x(t, k) = 0 \tag{23}$$

which implies that the unforced 2-D FM system Σ is 2-D-stable. □

Remark 3.2. 2-D-RCI quantifies the system 2-D-stability. $\rho > 1$ ensures the 2-D-stability and a larger 2-D-RCI indicates better convergence in the 2-D sense. For controller design, 2-D-RCI should be maximized.

Similar results can be obtained for T-stability and K-stability.

Theorem 3.2. The unforced 2-D system Σ is T-stable if there exist a function $V(\cdot)$ and a scalar $\alpha > 1$ satisfying.

(1) $V(x) \geq 0$ for $\forall x \in \mathbb{R}^n$, and $V(x) = 0 \Leftrightarrow x = 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$

(3) for any T-boundary $x_{0,k}$, integer $N > 0$, and zero K-boundary $x_{t,0}$,

$$\sum_{k=1}^{N} V(x(t+1, k)) < \alpha^{-1} \sum_{k=1}^{N} V(x(t, k)), \quad \forall\, t > 0 \tag{24}$$

Moreover, the maximum value α satisfying Eq. 24 is called *T robust convergence index* (T-RCI) of the system.

Theorem 3.3. The unforced 2-D system Σ is K-stable if there exist a function $V(\cdot)$ and a scalar $\beta > 1$ satisfying.

(1) $V(x) \geq 0$ for $\forall x \in \mathbb{R}^n$, and $V(x) = 0 \Leftrightarrow x = 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$

(3) for any K-boundary $x_{t,0}$, integer $N > 0$, and zero T-boundary $x_{0,k}$,

$$\sum_{t=1}^{N} V(x(t, k+1)) < \beta^{-1} \sum_{t=1}^{N} V(x(t, k)), \quad \forall\, k > 0 \tag{25}$$

Moreover, the maximum value β satisfying Eq. 25 is called *K robust convergence index* (K-RCI) of the system.

The proofs of the above theorems are straightforward and are thus omitted.

In Theorems 3.1–3.3, functions $V(\cdot)$ are analogous to the Lyapunov function for stability analysis of the 1-D system. We refer to them as *Lyapunov-like functions* for the 2-D system. The key to stability analysis of the 2-D system is to find a suitable Lyapunov-like function.

Theorem 3.4. For given scalars α, $\beta > 1$, the unforced 2-D FM system Σ is robustly stable if there exist PDS matrices P, Q_1, and $Q_2 \in \mathbb{R}^{n\times n}$ and scalars ε_1, $\varepsilon_2 > 0$ such that the following LMI conditions hold:

$$\begin{bmatrix} -P & PA_1 & PA_2 & PE_1 & PE_2 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 & \varepsilon_1 F_1^T & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & \varepsilon_2 F_2^T \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \tag{26}$$

$$\alpha Q_1 + \beta Q_2 < P \tag{27}$$

Moreover, T-RCI of the system is not less than α, K-RCI is not less than β, and 2-D-RCI is not less than $\rho = \min\{\alpha, \beta\}$.

The proof can be found in Appendix A1.

The following result is proposed for robust H_∞ performance analysis of a 2-D FM system.

Theorem 3.5. For given scalars $\gamma > 0$ and $\alpha, \beta > 1$, the unforced 2-D FM system Σ has robust H_∞ performance γ if there exist PDS matrices P, Q_1, and $Q_2 \in \mathbb{R}^{n\times n}$ and scalars ε_1, $\varepsilon_2 > 0$ such that the following LMI conditions hold:

$$\begin{bmatrix} -P & PA_1 & PA_2 & PE_1 & PE_2 & 0 & 0 & PH & 0 \\ * & -Q_1 & 0 & 0 & 0 & \varepsilon_1 F_1^T & 0 & 0 & C^T \\ * & * & -Q_2 & 0 & 0 & 0 & \varepsilon_2 F_2^T & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & * & * & * & -\gamma I \end{bmatrix} < 0 \quad (28)$$

$$\alpha Q_1 + \beta Q_2 < P \quad (29)$$

Moreover, T-RCI of the system is not less than α, K-RCI is not less than β, and 2-D-RCI is not less than $\rho = \min\{\alpha, \beta\}$.

The proof can be found in Appendix A2.

Remark 3.3. Sufficient conditions 26–27 and 28–29 are both expressed in LMI forms, which can be assessed by using LMI tools.[23]

Remark 3.4. From the proofs of Theorem 3.4 and Theorem 3.5, it can be seen that the given scalars α and β are, respectively, the possible lower bounds of T-RCI and K-RCI of the 2-D system. Tighter estimation of T-RCI (or K-RCI) can be obtained by regarding α (or β) as an unknown variable to be maximized, resulting in a standard Generalized Eigenvalue Problem (GEVP),[24] which can be solved by LMI tools.[23] Relationships among T-RCI, K-RCI, and the real convergence rates of the system are shown by inequality A19, where a larger α indicates definitely a better convergence along T-axis, whereas a larger β may also imply a faster convergence rate along the T-axis if the PDS matrices Q_1 and Q_2 are comparable. Similar conclusions can be drawn for the convergence along the K-axis.

Robust 2-D output feedback control

Because state feedback control is usually impracticable, it is more acceptable to consider the following dynamic 2-D output feedback controller for the 2-D FM system Σ:

$$\Sigma_c: \begin{cases} x_c(t+1, k+1) = A_{c1}x_c(t, k+1) + A_{c2}x_c(t+1, k) + B_{c1}y(t, k+1) + B_{c2}y(t+1, k) \\ u(t, k+1) = C_{c1}x_c(t, k+1) + C_{c2}x_c(t+1, k) + D_{c1}y(t, k+1) + D_{c2}y(t+1, k) \end{cases} \quad t, k = 0, 1, 2, \cdots \quad (30)$$

where $x_c(t, k) \in \mathbb{R}^n$ is the internal state of the controller and $\{A_{ci}, B_{ci}, C_{ci}, D_{ci}\}_{i=1,2}$ are controller parameters to be determined with appropriate dimensions.

Substituting Eq. 30 into system 12 results in the following closed-loop system:

$$\Sigma_{cl}: \begin{cases} \begin{bmatrix} x(t+1, k+1) \\ x_c(t+1, k+1) \end{bmatrix} = (\bar{A}_1 + \Delta\bar{A}_1)\begin{bmatrix} x(t, k+1) \\ x_c(t, k+1) \end{bmatrix} + (\bar{A}_2 + \Delta\bar{A}_2)\begin{bmatrix} x(t+1, k) \\ x_c(t+1, k) \end{bmatrix} + \bar{H}w(t, k+1) \\ y(t, k) = \bar{C}\begin{bmatrix} x(t, k) \\ x_c(t, k) \end{bmatrix} \\ z(t, k) = \bar{G}\begin{bmatrix} x(t, k) \\ x_c(t, k) \end{bmatrix} \end{cases} \quad (31)$$

where

$$\bar{A}_1 = \begin{bmatrix} A_1 + BD_{c1}C & BC_{c1} \\ B_{c1}C & A_{c1} \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} A_2 + BD_{c2}C & BC_{c2} \\ B_{c2}C & A_{c2} \end{bmatrix},$$

$$\bar{H} = \begin{bmatrix} H \\ 0 \end{bmatrix}, \quad \bar{C} = [C \quad 0], \quad \bar{G} = [G \quad 0]$$

$$\Delta\bar{A}_1 = \begin{bmatrix} \Delta A_1 + \Delta BD_{c1}C & \Delta BC_{c1} \\ 0 & 0 \end{bmatrix},$$

$$\Delta\bar{A}_2 = \begin{bmatrix} \Delta A_2 + \Delta BD_{c2}C & \Delta BC_{c2} \\ 0 & 0 \end{bmatrix} \quad (32)$$

The closed-loop system Σ_{cl} is an unforced 2-D FM system. Robust stabilization is to design a 2-D output feedback controller Σ_c such that the closed-loop system Σ_c is robustly stable with satisfied convergence indices. The results are as follows.

Theorem 3.6. For given scalars $\alpha, \beta > 1$, a sufficient condition for existing a 2-D output feedback controller Σ_c that guarantees the closed-loop 2-D system Σ_{cl} to be robustly stable with T-RCI not less than α, K-RCI not less than β, and 2-D-RCI not less than $\rho = \min\{\alpha, \beta\}$ is that there exist PDS matrices $X, Y \in \mathbb{R}^{n\times n}$, $S_1, S_2 \in \mathbb{R}^{2n\times 2n}$, matrices $\{\tilde{A}_{ci}, \tilde{B}_{ci}, \tilde{C}_{ci},$

$\tilde{D}_{ci}\}_{i=1,2}$ with appropriate dimensions and scalars $\{\varepsilon_i > 0\}_{i=1,2,3}$ such that the following matrix inequalities hold:

$$\begin{bmatrix} -\mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} & \mathbf{\Omega}_{13} & 0 \\ * & -\mathbf{\Omega}_{22} & 0 & \mathbf{\Omega}_{24} \\ * & * & -\mathbf{\Omega}_{33} & 0 \\ * & * & * & -\mathbf{\Omega}_{44} \end{bmatrix} < 0 \tag{33}$$

$$\alpha S_1 + \beta S_2 < \mathbf{\Omega}_{11} \tag{34}$$

where

$$\mathbf{\Omega}_{11} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \quad \mathbf{\Omega}_{22} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad \begin{matrix} \mathbf{\Omega}_{12} = [\mathbf{A}_1 \quad \mathbf{A}_2], \\ \mathbf{\Omega}_{13} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3], \end{matrix}$$

$$\mathbf{\Omega}_{24} = \begin{bmatrix} \mathbf{F}_1 & 0 & \mathbf{F}_{31} \\ 0 & \mathbf{F}_2 & \mathbf{F}_{32} \end{bmatrix}$$

$$\mathbf{A}_i = \begin{bmatrix} A_iX + B\tilde{C}_{ci} & A_i + B\tilde{D}_{ci}C \\ \tilde{A}_{ci} & YA_i + \tilde{B}_{ci}C \end{bmatrix}, \quad \mathbf{E}_j = \begin{bmatrix} I \\ Y \end{bmatrix} E_j,$$

$$\mathbf{F}_i = \begin{bmatrix} X \\ I \end{bmatrix} F_i^T, \quad \mathbf{F}_{3i} = \begin{bmatrix} \tilde{C}_{ci}^T \\ C^T\tilde{D}_{ci}^T \end{bmatrix} F_3^T$$

$$\mathbf{\Omega}_{33} = \mathrm{diag}\{\varepsilon_1 I, \varepsilon_2 I, \varepsilon_3 I\}, \quad \mathbf{\Omega}_{44} = \mathrm{diag}\{\varepsilon_1^{-1} I, \varepsilon_2^{-1} I, \varepsilon_3^{-1} I\}, \quad i = 1, 2, \quad j = 1, 2, 3 \tag{35}$$

If $\{X, Y, \tilde{A}_{ci}, \tilde{B}_{ci}, \tilde{C}_{ci}, \tilde{D}_{ci}\}_{i=1,2}$ represents a feasible solution of matrix inequalities 33 and 34, then a set of suitable parameters for 2-D output controller Σ_c can be constructed by

$$\begin{cases} D_{ci} = \tilde{D}_{ci} \\ C_{ci} = (\tilde{C}_{ci} - D_{ci}CX)M^{-T} \\ B_{ci} = N^{-1}(\tilde{B}_{ci} - YBD_{ci}) \\ A_{ci} = N^{-1}(\tilde{A}_{ci} - YA_iX - YB\tilde{C}_{ci})M^{-T} - B_{ci}CXM^{-T} \end{cases}, \quad i = 1, 2 \tag{36}$$

where full-rank matrices M and N, satisfying $XY + MN^T = I$, can be computed by the singular value decomposition of matrix $I - XY$.

The proof can be found in Appendix A3.

Robust H_∞ stabilization is to design a 2-D output feedback controller Σ_c such that the closed-loop system Σ_{cl} has specified robust H_∞ performance. The conclusion is as follows.

Theorem 3.7. For given scalars $\alpha, \beta > 1$, $\gamma > 0$, a sufficient condition for existing a 2-D output feedback controller Σ_c that guarantees the closed-loop 2-D system Σ_{cl} to have robust H_∞ performance γ, T-RCI not less than α, K-RCI not less than β, and 2-D-RCI not less than $\rho = \min\{\alpha, \beta\}$ is that there exist PDS matrices $X, Y \in \mathbb{R}^{n\times n}$, $S_1, S_2 \in \mathbb{R}^{2n\times 2n}$, matrices $\{\tilde{A}_{ci}, \tilde{B}_{ci}, \tilde{C}_{ci}, \tilde{D}_{ci}\}_{i=1,2}$ with appropriate dimensions and scalars $\{\varepsilon_i > 0\}_{i=1,2,3}$ such that the following matrix inequalities hold:

$$\begin{bmatrix} -\mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} & \mathbf{\Omega}_{13} & 0 & \mathbf{\Omega}_{15} \\ * & -\mathbf{\Omega}_{22} & 0 & \mathbf{\Omega}_{24} & \mathbf{\Omega}_{25} \\ * & * & -\mathbf{\Omega}_{33} & 0 & 0 \\ * & * & * & -\mathbf{\Omega}_{44} & 0 \\ * & * & * & * & -\mathbf{\Omega}_{55} \end{bmatrix} < 0 \tag{37}$$

$$\alpha S_1 + \beta S_2 < \mathbf{\Omega}_{11} \tag{38}$$

where $\mathbf{\Omega}_{11}$, $\mathbf{\Omega}_{12}$, $\mathbf{\Omega}_{13}$, $\mathbf{\Omega}_{22}$, $\mathbf{\Omega}_{24}$, $\mathbf{\Omega}_{33}$, $\mathbf{\Omega}_{44}$ are defined by Eq. 35, and $\mathbf{\Omega}_{15}$, $\mathbf{\Omega}_{25}$, $\mathbf{\Omega}_{55}$ are as follows:

$$\mathbf{\Omega}_{15} = \begin{bmatrix} H & 0 \\ YH & 0 \end{bmatrix}, \quad \mathbf{\Omega}_{25} = \begin{bmatrix} 0 & \begin{bmatrix} X \\ I \end{bmatrix} & G^T \\ 0 & 0 & \end{bmatrix}, \quad \mathbf{\Omega}_{55} = \begin{bmatrix} \gamma I & 0 \\ 0 & \gamma I \end{bmatrix} \tag{39}$$

If $\{X, Y, \tilde{A}_{ci}, \tilde{B}_{ci}, \tilde{C}_{ci}, \tilde{D}_{ci}\}_{i=1,2}$ represents a feasible solution of matrix inequalities 37 and 38, then a set of suitable parameters for 2-D output controller Σ_c can be constructed by formulas of Eq. 36, where full-rank matrices M and N, satisfying $XY + MN^T = I$, can be computed by the singular value decomposition of matrix $I - XY$.

The proof can be found in Appendix A3.

Remark 3.5. Note that sufficient conditions 33 and 37 are both not linear matrix inequalities because the variables $\{\varepsilon_i > 0\}_{i=1,2,3}$ enter in matrix $\mathbf{\Omega}_{44}$ in a nonlinear fashion. Thus, the available LMI tools cannot be used directly to obtain a feasible solution. The nonlinearity is encountered often in output feedback control in both 1-D and 2-D context.[13,27] An easy, but somewhat conservative way to deal with this problem is to fix the parameters $\{\varepsilon_i > 0\}_{i=1,2,3}$ in Eq. 33 or Eq. 37, which eliminates the nonlinear terms. If computational costs are not a problem, better results can be obtained by using iterative algorithms based on LMI convex optimization problems, as introduced briefly in Appendix A5.

Remark 3.6. The specification of parameters α, β, and γ reflects the designer's expectations of the T-RCI, K-RCI, and robust H_∞ performance of the control system. Ideally, α and β should be as large as possible, which guarantees the best convergences along both T-axis and K-axis, and γ should be as small as possible, leading to lower sensitivity to disturbance. The complicated and nonlinearity relationships among these three parameters, however, make it difficult to find the global optimum. Practically, a compromise or trade-off strategy should be used to find a suboptimal solution. A suboptimal algorithm introduced in the next section will be used in our design work.

System Structure and Design Algorithm

System structure

Based on the results of the last section, Design Objective 1 can be rephrased for the equivalent 2-D tracking error model Σ_p as follows.

Design Objective 2. For 2-D FM system Σ_p and all admissible uncertain parameter perturbations satisfying norm-bounded conditions (Eq. 3), design a 2-D output feedback controller Σ_c such that

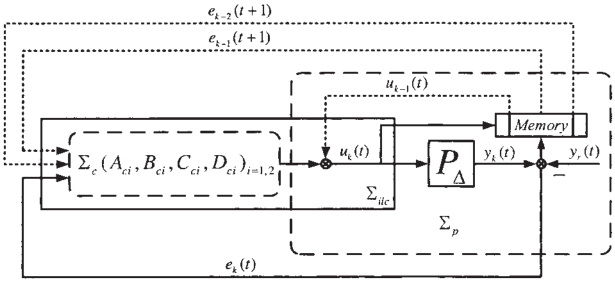


Figure 1. Block diagram of ILC system (solid line blocks) or 2-D system (dash line blocks).

- the closed-loop system is 2-D-stable
- the closed-loop system is T-stable with T-RCI as large as possible
- the closed-loop system is K-stable with K-RCI as large as possible
- if nonrepeatable disturbance $w(t, k)$ is nonzero, the robust H_∞ performance of the closed-loop system should be as small as possible

Design Objective 2 indicates that the design of ILC law 4 can be cast as a robust 2-D output feedback control of 2-D FM system Σ_p. If the above design objective can be achieved, then the updating law $r(t, k)$ can be interpreted by 2-D system Σ_c and the resulting ILC system can be depicted by Figure 1, where the dot arrow lines represent the flow of the information of previous cycles, whereas the solid arrow lines represent the flow of the real time feedback information. This block diagram figure can be interpreted from two different angles: one is 2-D system, another is the ILC system. From the perspective of the 2-D system, it is a 2-D output feedback control system consisting of 2-D plant Σ_p and 2-D output feedback controller Σ_c, as shown by the dash line blocks with round corners. From the ILC system perspective, the control system consists of a plant P_Δ and an ILC law Σ_{ilc}, as shown by solid line blocks. It can be clearly seen from the block diagram that information, real-time feedback information, and two tracking errors of last two cycles, are used as the learning information, which implies the action of the ILC law comes from two sources: one is feedback control, another is high-order feedforward ILC. Feedback control guarantees the robustness and control performance along the time direction, whereas the high-order feedforward ILC improves the control accuracy along the cycle direction. The two controllers are designed in an integrated fashion within the framework of 2-D system, resulting in the actions integrated harmoniously in a 2-D sense.

Design algorithm

According to the repeatability of the parameter perturbations, Theorem 3.6 or Theorem 3.7 should be applied for the ILC design. For simplicity, in this section we take the plant with nonrepeatable parameter perturbations as an example.

Because the nonrepeatable disturbance w is nonzero, Theorem 3.7 should be applied, and the iterative algorithm introduced in Appendix A5 can be used to solve the nonlinear inequalities 37 and 38 when the performance indices α, β, and γ are specified. It is reasonable to regard all three indices as the design parameters to be optimized. As mentioned earlier, however, it is difficult to optimize all three indices simultaneously. Practically, we can optimize these design parameters according to the specified priorities that reflect the relative importance of these indices. The idea is summarized as the following algorithm.

Algorithm

Step 1. Guaranteed performance design: Given a set of performance indices $\alpha > 1$, $\beta > 1$, and $\gamma > 0$ that must be satisfied, referred to as guaranteed performance indices, solve matrix inequalities 37 and 38 proposed in Theorem 3.7 by using an iterative algorithm (see Appendix A5). If there is a feasible solution, then use these indices as the initial indices and go to next step; otherwise, the guaranteed performance indices should be adjusted.

Step 2. Prioritized optimization: Set up priorities for the three indices according to their relative practical importance. For example, let K-RCI, β, have the top priority, indicating that the convergence rate along the cycle direction is the most important in the control performance. By using a fixed-step or variable-step search algorithm, optimize the performance index of the highest priority with fixed indices for other priorities until the optimal (or satisfactory) value is obtained. After that, switch to optimize the performance index with the secondary priority with the other indices fixed. Repeat the procedure until all performance indices have optimal (or satisfactory) values.

Step 3. Controller parameter computation: With $\{X, Y, \tilde{A}_{ci}, \tilde{B}_{ci}, \tilde{C}_{ci}, \tilde{D}_{ci}\}_{i=1,2}$ as the final feasible solution of the last step, compute the controller parameters $\{A_{ci}, B_{ci}, C_{ci}, D_{ci}\}_{i=1,2}$ according to formulas of Eq. 36, where the full-rank matrices M and N, which satisfy $XY + MN^T$, can be computed by the singular value decomposition of matrix $I - XY$.

Extension of Learning Information

For the ILC scheme, use of more learning information may lead to a better control performance. An advantage of using the design framework proposed in this article is that the learning information used by the ILC law can be flexibly extended, along the time and/or the cycle, leading to a high-order ILC scheme. Use of the extended learning information for control performance improvement is determined by the 2-D system Σ_c, which is synthesized directly by the proposed design algorithm.

Consider the following general extension of 2-D dynamical

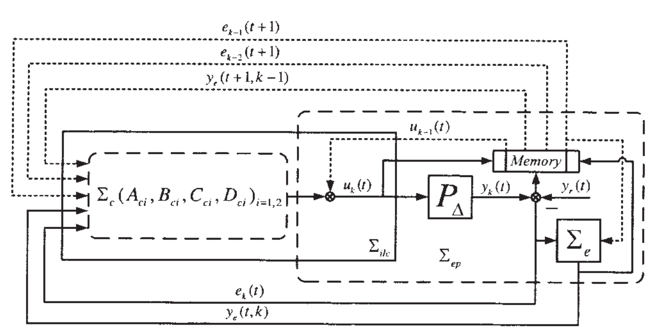


Figure 2. Block diagram of ILC system with extended learning information (solid line blocks) or 2-D system (dash line blocks).

model for incorporating more learning information for the ILC scheme.

$$\sum_e \begin{cases} x_e(t+1,k) = A_{e1}x_e(t,k) + A_{e2}x_e(t+1,k-1) \\ \quad + B_{e1}e(t,k) + B_{e2}e(t+1,k-1) \\ y_e(t,k) = C_e x_e(t,k) + D_e e(t,k) \end{cases} \tag{40}$$

where $x_e(t, k) \in \mathbb{R}^q$ is the state of the extension model, $\{A_{ei}, B_{ei}, C_e, D_e\}_{i=1,2}$ are the matrices of proper dimensions that are specified for the structure of the extended learning information. For example, to eliminate the steady-state tracking error resulting from nonrepeatable parameter perturbation, as seen in the next section, the integral of tracking error along time can be added as the extended 2-D model state for ILC design. In this case, the specification that $A_{e1} = B_{e1} = C_e = D_e = I$ and $A_{e2} = B_{e2} = 0$ results in Σ_e being an integrator along the time. The specification of structural matrices allows flexible assembling of tracking errors as the extended learning information to enhance the system performance along the time and/or the cycle directions.

The combination of 2-D models Σ_e and Σ_p leads to an augmented 2-D model as follows:

$$\Sigma_{ep}: \begin{cases} \begin{bmatrix} \Delta x(t+1,k) \\ x_e(t+1,k) \\ e(t+1,k) \end{bmatrix} = (\hat{A}_1 + \hat{\Delta}_a(t,k)) \begin{bmatrix} \Delta x(t,k) \\ x_e(t,k) \\ e(t,k) \end{bmatrix} + \hat{A}_2 \begin{bmatrix} \Delta x(t+1,k-1) \\ x_e(t+1,k-1) \\ e(t+1,k-1) \end{bmatrix} + (\hat{B} + \hat{\Delta}_b(t,k))r(t,k) + \hat{H}w(t,k) \\ y(t,k) \triangleq \begin{bmatrix} e(t,k-1) \\ y_e(t,k) \\ e(t,k) \end{bmatrix} = \hat{C} \begin{bmatrix} \Delta x(t,k) \\ x_e(t,k) \\ e(t,k) \end{bmatrix} \\ z(t,k) = \hat{G} \begin{bmatrix} \Delta x(t,k) \\ x_e(t,k) \\ e(t,k) \end{bmatrix} \end{cases} \tag{41}$$

where

$$\hat{A}_1 = \begin{bmatrix} A & 0 & 0 \\ 0 & A_{e1} & B_{e1} \\ CA & 0 & 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{e2} & B_{e2} \\ 0 & 0 & I \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} B \\ 0 \\ CB \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} I \\ 0 \\ C \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} -C & 0 & I \\ 0 & C_e & D_e \\ 0 & 0 & I \end{bmatrix}$$

$$\hat{G} = [0 \quad 0 \quad I], \quad \hat{\Delta}_a = \begin{bmatrix} \Delta_a(t,k) & 0 & 0 \\ 0 & 0 & 0 \\ C\Delta_a(t,k) & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_a \\ 0 \\ CE_a \end{bmatrix} \Delta_1(t,k)[F_a \quad 0 \quad 0] \triangleq \hat{E}_a \Delta_1(t,k) \hat{F}_a$$

$$\hat{\Delta}_b = \begin{bmatrix} \Delta_b(t,k) \\ 0 \\ C\Delta_b(t,k) \end{bmatrix} = \begin{bmatrix} E_b \\ 0 \\ CE_b \end{bmatrix} \Delta_2(t,k) F_b \triangleq \hat{E}_b \Delta_2(t,k) \hat{F}_b \tag{42}$$

Obviously, the augmented 2-D model Σ_{ep} provides more output information resulting in the obtained 2-D updating law Σ_c with a higher order. Figure 2 depicts the block diagram of the closed-loop system with the extended learning information.

Examples

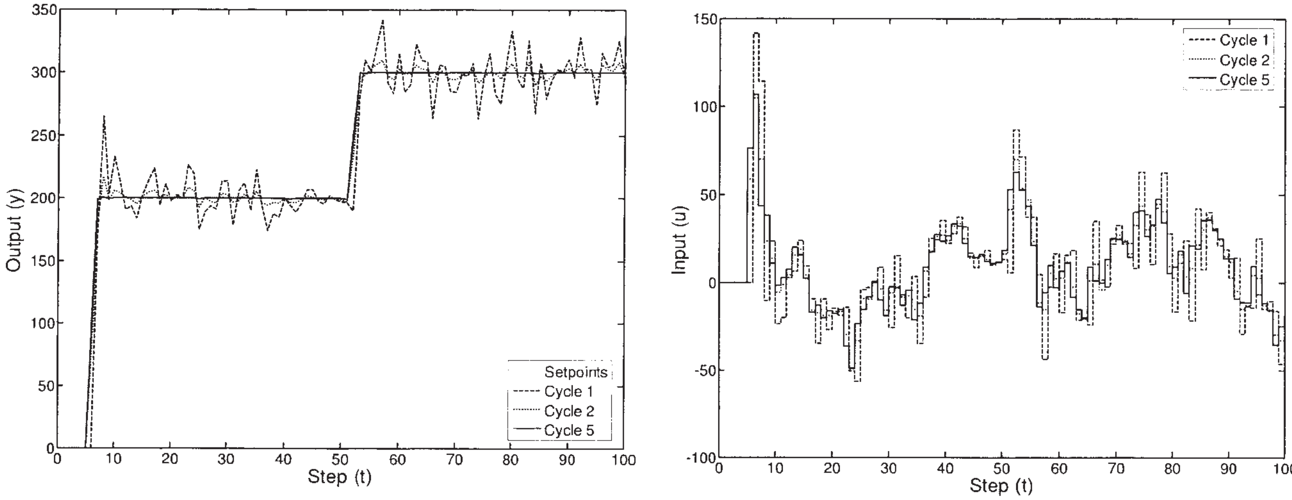
Injection molding, a cyclic process, consists of three main stages: filling, packing-holding, and cooling.[25] For the packing-holding stage, nozzle pressure is a key process variable that should be controlled to follow a certain profile to ensure product quality. Variations of working conditions, including mold, material, and dynamical behavior of hydraulic actor, mean that injection molding, particularly packing-holding, should be viewed as a batch process with uncertain parameter perturbations. In each cycle, switch of filling to packing-holding, referred to as V/P transfer, leads to an unpredictable initial value of the nozzle pressure. This makes the conventional ILC, such as P-type ILC, not applicable. On the other hand, the control performance is typically poor when a slow hydraulic valve is used. Pure feedback control, such as PID (proportional integrative derivative) control and adaptive control,[26] cannot improve control performance from cycle to cycle.

The proposed ILC scheme, inherently an integration of both feedback and iterative controls, is applied to control the nozzle packing pressure to demonstrate the effectiveness of the proposed method.

Table 1. Design Results for Case 1 ($\alpha = 1.01 \quad \beta^* = 3.16$)

	$i = 1$	$i = 2$
A_{ci}	$\begin{bmatrix} -0.2159 & 0.0000 & 0.0000 \\ -3.4055 & 0.8852 & 0.0000 \\ 7.5611 & -0.9896 & -0.4004 \end{bmatrix}$	$\begin{bmatrix} 0.0593 & 0.0000 & 0.0000 \\ 0.1833 & 0.0000 & 0.0000 \\ 7.0258 & -0.0731 & -0.0383 \end{bmatrix}$
B_{ci}	$\begin{bmatrix} 0.1444 & 0.0000 \\ 1.4388 & -1.0003 \\ -8.0551 & 6.4082 \end{bmatrix}$	$\begin{bmatrix} -0.0396 & 0.6688 \\ -0.1226 & 1.3026 \\ -4.7070 & -2.6201 \end{bmatrix}$
C_{ci}	$[-2.0735 \quad 0.6704 \quad 0.0000]$	$[0.0000 \quad 0.0000 \quad 0.0000]$
D_{ci}	$[1.4072 \quad -1.4133]$	$[0.0000 \quad -0.5870]$

Note: * optimized variable.



(a) (b)

Figure 3. Simulation results of Case 1 without initial errors.

(a) Output responses; (b) control inputs.

Model

Based on the open-loop tests and analysis, the model of the nozzle packing pressure response to the hydraulic control valve opening can be identified as

$$P_\Delta(z^{-1}) = \frac{1.239(0.10)z^{-1} - 0.9282(0.14)z^{-2}}{1 - 1.607(0.08)z^{-1} + 0.6086(0.08)z^{-2}} \quad (43)$$

where the numbers in the brackets show the typical norm bounds of the parameter perturbations. As an economical proportional valve, rather than the expensive servo valve, is used, the nominal model has quite slow response characteristic.

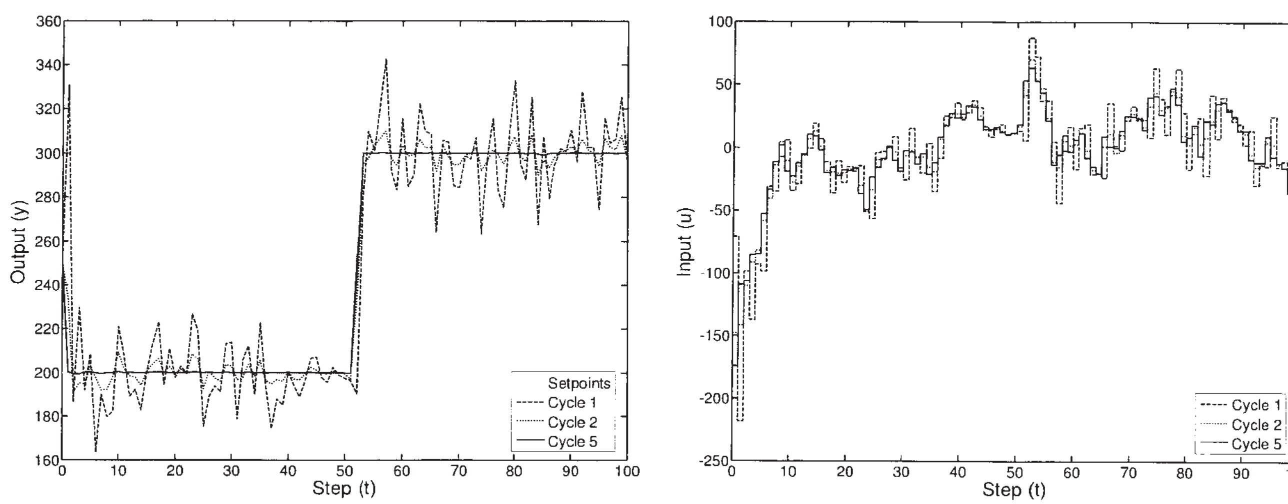
The model is converted into uncertain state–space representation as follows:

$$P_\Delta: \begin{cases} x(t+1) = \left(\begin{bmatrix} 1.607 & 1 \\ -0.6086 & 0 \end{bmatrix} + \Delta_a\right)x(t) \\ \quad + \left(\begin{bmatrix} 1.2390 \\ -0.9282 \end{bmatrix} + \Delta_b\right)u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases} \quad (44)$$

where the uncertain parameter perturbations can be represented as

$$\Delta_a = E_a\Delta_1 F_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Delta_1\begin{bmatrix} 0.08 & 0 \\ 0.08 & 0 \end{bmatrix},$$

$$\Delta_b = E_b\Delta_2 F_b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Delta_2\begin{bmatrix} 0.10 \\ 0.14 \end{bmatrix} \quad (45)$$



(a) (b)

Figure 4. Simulation results of Case 1 with fixed initial errors.

(a) Output responses; (b) control inputs.

Table 2. Design Results for Case 2 without the Use of Integral of Tracking Error as the Extended Learning Information ($\alpha^* = 1.02 \quad \beta = 1.01 \quad \gamma^{**} = 14.89$)

	$i = 1$	$i = 2$
A_{ci}	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ -0.3486 & 0.9358 & 0.0000 \\ -0.2821 & -2.6862 & -0.2902 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 4.5592 & -0.0204 & -0.0984 \end{bmatrix}$
B_{ci}	$\begin{bmatrix} 0.0000 & 0.0000 \\ 0.2221 & -0.2391 \\ -8.2348 & 7.9625 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.3540 \\ 0.0000 & 0.0923 \\ 0.0811 & -2.2288 \end{bmatrix}$
C_{ci}	$[-1.1678 \quad 3.1354 \quad 0.0000]$	$[0.0000 \quad 0.0000 \quad 0.0000]$
D_{ci}	$[1.3999 \quad -1.4568]$	$[0.0000 \quad -0.1932]$

Note: * primary optimized variable; ** secondary optimized variable.

$$\Delta_1 = \text{diag}\{\delta_{11}(t), \delta_{12}(t)\}, \quad \Delta_2 = \text{diag}\{\delta_{21}(t), \delta_{22}(t)\} \quad (46)$$

where $\{\delta ij : |\delta_{ij}| < 1\}_{i,j=1,2}$ denote the unknown perturbations.

Designs and simulations

To illustrate the abilities of the proposed scheme in handling initial condition problem and uncertain parameter perturbations along time and cycle directions, three cases are simulated.

Case 1: Repeatable Perturbations. In this case, nonrepeatable disturbance w is zero. Based on Theorem 3.6, the design algorithm proposed in the 4th section is conducted with K-RCI, β, taken as the only index to be optimized, whereas the T-RCI is fixed at 1.01. Table 1 gives the design results.

To test the robustness of control system to the repeatable perturbations, $\{\delta_{ij}(t) < 1\}_{i,j=1,2}$ are taken as repeatable random variables distributed within $[-1, 1]$. Figure 3 shows the output responses and the control inputs of the process without initial tracking error. It is clearly seen that the closed-loop system has a good robustness to repeatable perturbations and the noticeable tracking error in the first cycle converges rapidly to zero along the cycle direction. Figure 4 gives the control results with a fixed initial error. It is clearly observed that initial tracking error does not destroy the ILC performance, and that the tracking errors converge rapidly to zeros along both the cycle and time directions.

Case 2: Nonrepeatable Perturbations. In this case, the robustness along time and the disturbance rejection to nonrepeatable disturbance are the focus of concern. In the design procedure, T-RCI, α, is first optimized, then the H_∞ performance γ, whereas K-RCI, β, is fixed at 1.01. Table 2 gives the design results.

In simulation, $\{\delta_{ij}(k) < 1\}_{i,j=1,2}$ are assumed to be a cycle-varying random variable distributed within $[-1, 1]$. The control results with a fixed initial error are shown in Figure 5. Although robust stabilities along both the time and the cycle directions are guaranteed, nonrepeatable disturbance results in uncertain steady-state tracking errors within each cycle. To improve the control performance along the time direction, timewise integral of tracking errors need to be extended as the additional learning information. Based on the extension method proposed in the 5th section, a new ILC law that uses this new learning information was obtained and given in Table 3. It is seen from the control results, as shown in Figure 6, that the control performance of each cycle is improved along the time direction.

Case 3: Random Perturbations. Practically, perturbations can be divided into two parts, repeatable and nonrepeatable. In this case, it is assumed that total perturbations are the weighted sum of the two parts, expressed as follows:

$$\delta_{ij}(t, k) = 0.5\delta'_{ij}(t) + 0.5\delta''_{ij}(k) \qquad i, j = 1, 2 \quad (47)$$

where $\{\delta'_{ij}(t), \delta''(k)_{ij}\}_{i,j=1,2}$ are random variables distributed within $[-1, 1]$, representing the repeatable part and nonrepeatable part, respectively. The integral of tracking errors along time is also extended as additional learning information for

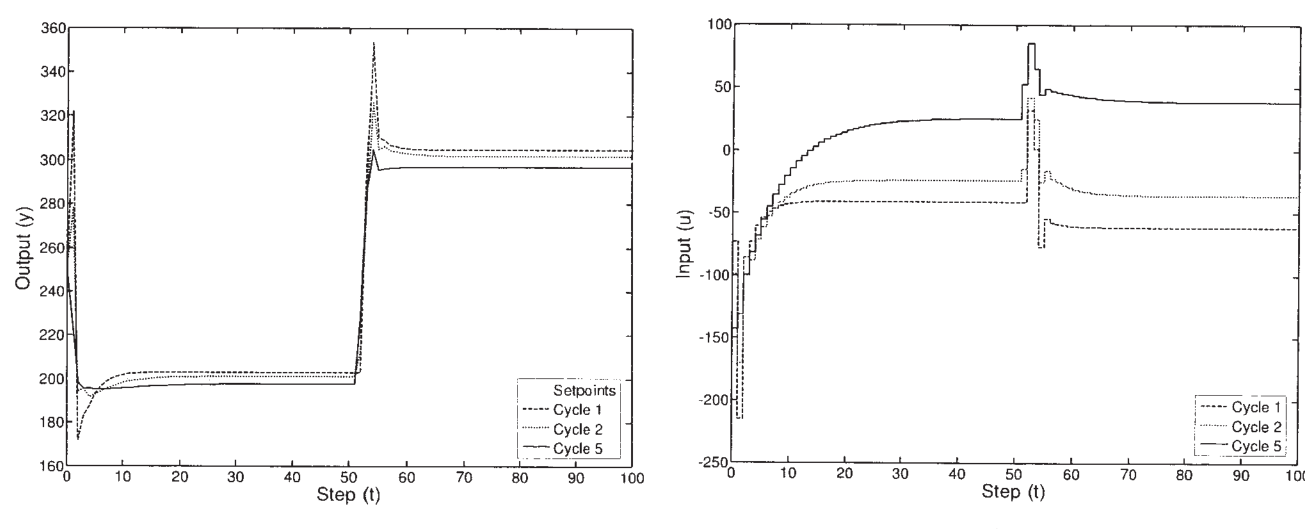


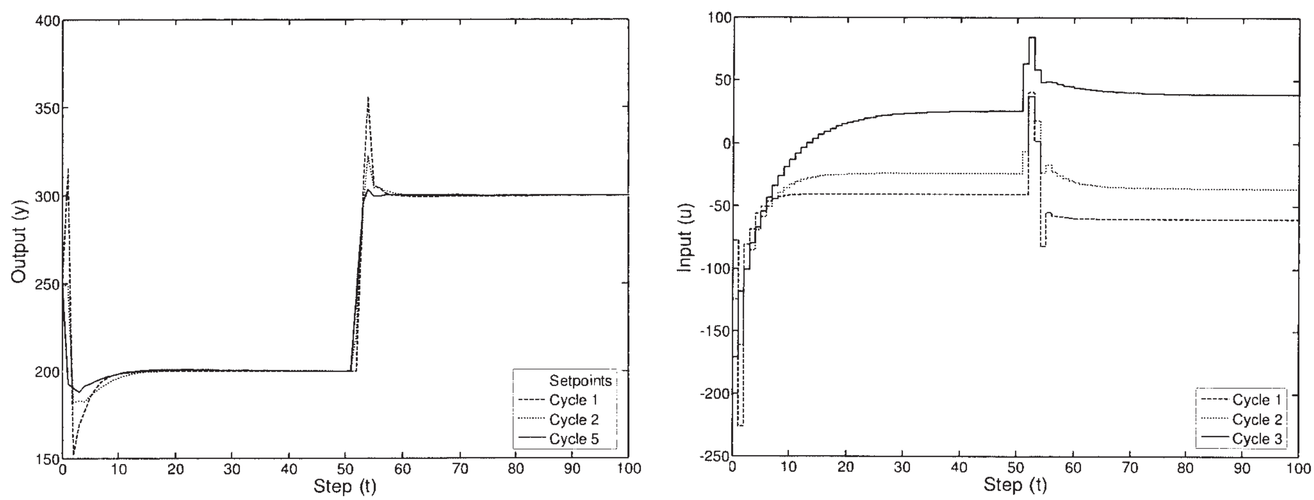
Figure 5. Simulation results of Case 2 without the use of integral of tracking error as the extended learning information.

(a) Output responses; (b) control inputs.

Table 3. Design Results for Case 2 with the Use of Integral of Tracking Error as the Extended Learning Information ($\alpha^* = 1.04 \quad \beta = 1.01 \quad \gamma^{**} = 29.98$)

	$i = 1$	$i = 2$
A_{ci}	$\begin{bmatrix} 0.1576 & 0.0000 & 0.0000 & 0.0000 \\ 0.0477 & -0.0005 & 0.0000 & 0.0000 \\ -0.6827 & -0.8441 & 0.9677 & 0.0000 \\ -1.0823 & -0.2940 & 1.6122 & 0.0417 \end{bmatrix}$	$\begin{bmatrix} 0.0002 & 0.0000 & 0.0000 & 0.0000 \\ -1.4146 & 0.0003 & 0.0000 & 0.0000 \\ -1.2330 & 0.0003 & 0.0000 & 0.0000 \\ -0.5670 & 0.0009 & 0.0002 & 0.0005 \end{bmatrix}$
B_{ci}	$\begin{bmatrix} 0.0000 & -0.0867 & -0.0162 \\ -0.0004 & 0.0076 & -0.0049 \\ -0.4082 & -0.0358 & 0.4335 \\ 0.3832 & -0.0640 & 0.5012 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & -0.0001 \\ 0.0001 & -0.1456 & -0.6686 \\ 0.0001 & -0.1269 & -0.4297 \\ -0.0958 & -0.0489 & 0.0097 \end{bmatrix}$
C_{ci}	$[1.6266 \quad 1.5860 \quad -1.8193 \quad 0.0000]$	$[0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000]$
D_{ci}	$[1.4224 \quad -0.0434 \quad -1.5059]$	$[0.0000 \quad 0.0000 \quad -0.3519]$

Note: * primary optimized variable; ** secondary optimized variable.



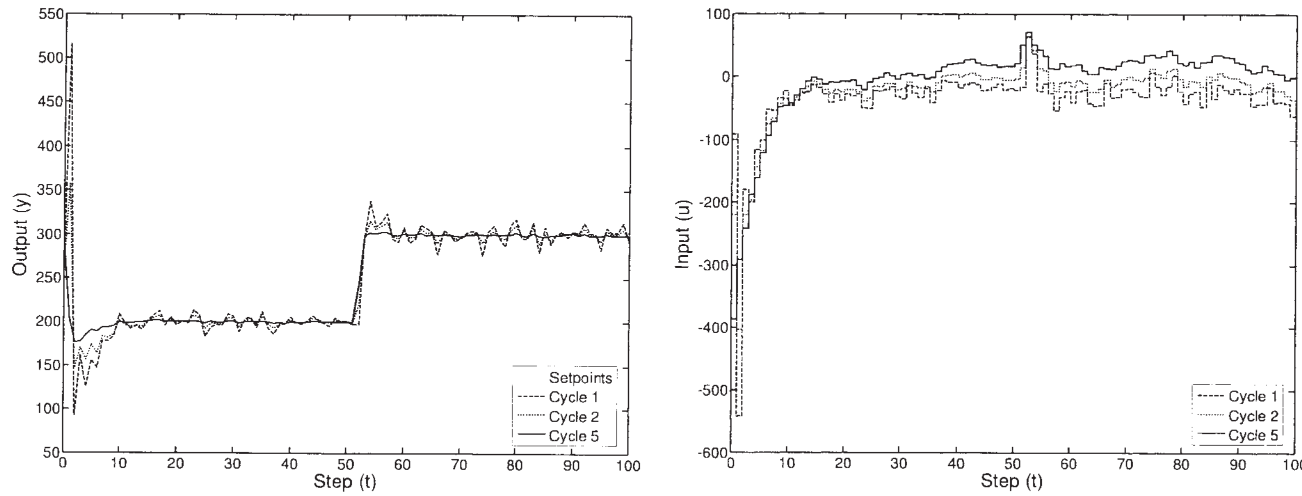
(a) (b)

Figure 6. Simulation results of Case 2 with the use of integral of tracking error as the extended learning information.
(a) Output responses; (b) control inputs.

Table 4. Design Results for Case 3 ($\alpha = 1.03 \quad \beta^* = 1.21 \quad \gamma^{**} = 29.98$)

	$i = 1$	$i = 2$
A_{ci}	$\begin{bmatrix} 0.0001 & 0.0000 & 0.0000 & 0.0000 \\ 0.0009 & 0.0000 & 0.0000 & 0.0000 \\ -0.5849 & -0.9987 & 0.9735 & 0.0000 \\ -1.6426 & -1.5709 & 1.3162 & 0.0934 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0012 & 0.0000 & 0.0000 & 0.0000 \\ -0.0013 & 0.0000 & 0.0000 & 0.0000 \\ -0.2813 & 0.7217 & 0.1288 & -0.0830 \end{bmatrix}$
B_{ci}	$\begin{bmatrix} 0.0000 & -0.0857 & 0.0000 \\ 0.0000 & -0.0009 & -0.0001 \\ -0.4419 & -0.0215 & 0.4809 \\ -6.1036 & 1.2098 & 8.6336 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0001 \\ 0.0000 & -0.0001 & -0.7499 \\ 0.0000 & -0.0001 & -0.5938 \\ -0.1319 & 0.0564 & -7.2933 \end{bmatrix}$
C_{ci}	$[1.0162 \quad 1.7322 \quad -1.6885 \quad 0.0000]$	$[0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000]$
D_{ci}	$[1.4221 \quad -0.0530 \quad -1.4899]$	$[0.0000 \quad 0.0000 \quad -0.3730]$

Note: * primary optimized variable; ** secondary optimized variable.



(a) (b)

Figure 7. Simulation results of Case 3.

(a) Output responses; (b) control inputs.

control performance enhancement. Table 4 gives the design result with the K-RCI, β, taken as the primary optimized variable, H_∞ performance γ taken as the secondary optimized variable, whereas T-RCI is fixed at 1.03. To test the robustness of the control system to the uncertain initialization, the initial conditions of each cycle are taken as a random number distributed within interval [200, 300]. The control results, as shown in Figure 7, illustrate that the proposed ILC scheme can effectively handle not only perturbations along the time and cycle but also the uncertain initial errors.

Conclusions

This article treats the ILC system as a 2-D system where the process and the feedback control govern the dynamical behavior along time direction, whereas the feedforward ILC law introduces dynamics along the cycle direction. Based on this, the convergences of the ILC system along the time and cycle directions have been defined and investigated from a 2-D system perspective. The design methods proposed in the framework of the 2-D system result in the integration of a feedback control and a feedforward ILC. The resulted ILC scheme can control effectively batch processes with both uncertain parameter perturbations and nonzero initial errors. Using our proposed design framework, the feedback and feedforward controls of the ILC system can be flexibly enhanced by extending the learning information, such as error integral, as illustrated in the article. All analysis and design methods are formulated as matrix inequality conditions that can be solved by LMI-based algorithm.

Acknowledgments

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Appendix

The following matrix inequalities are needed for the proofs of Theorems 3.4–3.7.

Lemma A1[24] Assume X, Y are matrices or vectors with appropriate dimensions. For any scalar $\varepsilon > 0$ and all matrices Δ with appropriate dimensions satisfying $\Delta\Delta^T \le I$, the following inequality holds:

$$X\Delta Y + Y^T\Delta^T X^T \le \varepsilon XX^T + \varepsilon^{-1}Y^TY \tag{A1}$$

Lemma A2[10] Assume A, E, F, and $Q = Q^T$ are given matrices with appropriate dimensions. For all matrices Δ, satisfying $\Delta\Delta^T \le I$, there exists a PDS matrix P satisfying

$$(A + E\Delta F)^T P(A + E\Delta F) - Q < 0 \tag{A2}$$

if and only if there exist a scalar $\varepsilon > 0$ and a PDS matrix P such that

$$\begin{bmatrix} -Q + \varepsilon F^TF & A^TP & 0 \\ PA & -P & PE \\ 0 & E^TP & -\varepsilon I \end{bmatrix} < 0 \tag{A3}$$

Lemma A3[24] (*Schur Complement*) Assume W, L, and V are given matrices with appropriate dimensions, where W and V are PDS matrices. Then

$$L^TVL - W < 0 \tag{A4}$$

if and only if

$$\begin{bmatrix} -W & L^T \\ L & -V^{-1} \end{bmatrix} < 0, \tag{A5}$$

or

$$\begin{bmatrix} -V^{-1} & L \\ L^T & -W \end{bmatrix} < 0 \tag{A6}$$

A1. Proof of Theorem 3.4

Proof: Defining the following quadratic function

$$V_{[\cdot]}(x(t, k)) = \|x(t, k)\|_{[\cdot]} \triangleq x^T(t, k)[\,\cdot\,]x(t, k) \tag{A7}$$

where $[\,\cdot\,]$ represents any PDS/SPDS matrix with appropriate dimensions. For PDS matrices P, Q_1, and Q_2, all functions $V_P(\cdot)$, $V_{Q_1}(\cdot)$, and $V_{Q_2}(\cdot)$ satisfy conditions (1) and (2) of Theorems 3.1, 3.2, and 3.3, respectively.

Now, assume $w \equiv 0$ and $u \equiv 0$. For an unforced 2-D FM system Σ, we have

$$\begin{aligned} \|x(t+1, k+1)\|_P - \|x(t, k+1)\|_{Q_1} - \|x(t+1, k)\|_{Q_2} &= \left\| \begin{bmatrix} x(t, k+1) \\ x(t+1, k) \end{bmatrix} \right\|_{\begin{bmatrix} A_1^T+\Delta A_1^T \\ A_2^T+\Delta A_2^T \end{bmatrix} P[A_1+\Delta A_1 \quad A_2+\Delta A_2]} - \left\| \begin{bmatrix} x(t, k+1) \\ x(t+1, k) \end{bmatrix} \right\|_{\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}} \\ &= \left\| \begin{bmatrix} x(t, k+1) \\ x(t+1, k) \end{bmatrix} \right\|_{\left(\left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} + \begin{bmatrix} F_1^T & 0 \\ 0 & F_2^T \end{bmatrix}\begin{bmatrix} \Delta_1^T & 0 \\ 0 & \Delta_2^T \end{bmatrix}\begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix}\right) P \left([A_1 \quad A_2] + [E_1 \quad E_2]\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\right) - \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\right)} \end{aligned} \tag{A8}$$

By applying Lemma A2 and Lemma A3, the following inequality can be derived from condition 26:

$$\left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} + \begin{bmatrix} F_1^T & 0 \\ 0 & F_2^T \end{bmatrix}\begin{bmatrix} \Delta_1^T & 0 \\ 0 & \Delta_2^T \end{bmatrix}\begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix}\right) P \Big([A_1 \quad A_2] + [E_1 \quad E_2]\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\Big) - \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} < 0 \tag{A9}$$

As the result, it follows that

$$\|x(t+1, k+1)\|_P < \|x(t, k+1)\|_{Q_1} + \|x(t+1, k)\|_{Q_2} \tag{A10}$$

Without loss of generality, suppose $\rho = \alpha = \min\{\alpha, \beta\}$. It results from condition 27 that $Q_1 < \rho^{-1}P - Q_2$, followed by

$$\|x(t+1, k+1)\|_P < \rho^{-1}\|x(t, k+1)\|_P - \|x(t, k+1)\|_{Q_2} + \|x(t+1, k)\|_{Q_2} \tag{A11}$$

Thus, for any integers T_0, K_0, $i > 0$, the following inequalities hold:

$$\|x(T_0+1, K_0+i)\|_P < \rho^{-1}\|x(T_0, K_0+i)\|_P - \|x(T_0, K_0+i)\|_{Q_2} + \|x(T_0+1, K_0+i-1)\|_{Q_2} \quad (A12)$$

$$\|x(T_0+2, K_0+i-1)\|_P < \rho^{-1}\|x(T_0+1, K_0+i-1)\|_P - \|x(T_0+1, K_0+i-1)\|_{Q_2} + \|x(T_0+2, K_0+i-2)\|_{Q_2} \quad (A13)$$

. . .. . .

$$\|x(T_0+i, K_0+1)\|_P < \rho^{-1}\|x(T_0+i-1, K_0+1)\|_P - \|x(T_0+i-1, K_0+1)\|_{Q_2} + \|x(T_0+i, K_0)\|_{Q_2} \quad (A14)$$

The sum of these inequalities leads to the following result

$$\sum_{\substack{t+k=T_0+K_0+i+1\\ T_0\le t\le T_0+i\\ K_0\le k\le K_0+i}} \|x(t, k)\|_P < \rho^{-1} \sum_{\substack{t+k=T_0+K_0+i\\ T_0\le t\le T_0+i\\ K_0\le k\le K_0+i}} \|x(t, k)\|_P - \|x(T_0, K_0+i)\|_{Q_2} + \|x(T_0+i, K_0)\|_{Q_2} - \rho^{-1}\|x(T_0+i, K_0\|_P < \rho^{-1} \sum_{\substack{t+k=T_0+K_0+i\\ T_0\le t\le T_0+i\\ K_0\le k\le K_0+i}} \|x(t, k)\|_P \quad (A15)$$

that is, condition (3) of Theorem 3.1 is satisfied, which implies that the 2-D FM system Σ is 2-D-stable with 2-D-RCI not less than ρ.

The proof of T-RCI not less than α is given as follows.

Obviously, the following inequalities hold according to Eq. A10 and condition 27:

$$\alpha\|x(t+1, N)\|_{Q_1} + \beta\|x(t+1, N)\|_{Q_2} < \|x(t, N)\|_{Q_1} + \|x(t+1, N-1)\|_{Q_2} \quad (A16)$$

$$\alpha\|x(t+1, N-1)\|_{Q_1} + \beta\|x(t+1, N-1)\|_{Q_2} < \|x(t, N-1)\|_{Q_1} + \|x(t+1, N-2)\|_{Q_2} \quad (A17)$$

. . .. . .

$$\alpha\|x(t+1, 1)\|_{Q_1} + \beta\|x(t+1, 1)\|_{Q_2} < \|x(t, 1)\|_{Q_1} + \|x(t+1, 0)\|_{Q_2} \quad (A18)$$

From the assumption that K-boundary conditions are zeros and condition $\beta > 1$, it follows that

$$\alpha \sum_{k=1}^{N} \|x(t+1, k)\|_{Q_1} < \alpha \sum_{k=1}^{N} \|x(t+1, k)\|_{Q_1} + \beta\|x(t+1, N)\|_{Q_2} + (\beta-1) \sum_{k=1}^{N-1} \|x(t+1, k)\|_{Q_2} < \sum_{k=1}^{N} \|x(t, k)\|_{Q_1} \quad (A19)$$

that is, condition (3) of Theorem 3.2 is satisfied. This completes the proof that the T-RCI of the 2-D FM system is not less than α. For the conclusion that K-RCI of the 2-D FM system is not less than β, the proof is similar and is therefore omitted. Thus, the proof of Theorem 3.4 is completed. □

A2. Proof of Theorem 3.5

Proof: According to matrix theory, Eq. 28) ⇒ Eq. 26, which means the 2-D FM system is robustly stable with T-RCI and K-RCI not less than α and β, respectively, and 2-D-RCI not less than $\rho = \min\{\alpha, \beta\}$. It remains to be proved that the 2-D FM system has robust H_∞ performance γ.

Choose quadratic function

$$J(t, k) = \Delta V(t, k) + \gamma^{-1}\|x(t, k)\|_{GG^T} - \gamma\|w(t, k)\|_I \quad (A20)$$

where

$$\Delta V(t, k) = \|x(t+1, k)\|_P - \|x(t, k)\|_{Q_1} - \|x(t+1, k-1)\|_{Q_2} \quad (A21)$$

According to dynamic Eq. 12, we have

$$J(t, k) = \left\| \begin{bmatrix} x(t, k) \\ x(t+1, k-1) \\ w(t, k) \end{bmatrix} \right\|_{\left(\begin{bmatrix} A_1^T+F_1^T\Delta_1^TE_1^T \\ A_2^T+F_2^T\Delta_2^TE_2^T \\ H^T \end{bmatrix} P[A_1+E_1\Delta_1F_1 \quad A_2+E_2\Delta_2F_2 \quad H] - \begin{bmatrix} Q_1-\gamma^{-1}G^TG & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \gamma I \end{bmatrix}\right)} \quad (A22)$$

By applying Lemmas A1–A3, the following inequality can be derived from matrix inequality 28:

$$\begin{bmatrix} A_1^T+F_1^T\Delta_1^TE_1^T \\ A_2^T+F_2^T\Delta_2^TE_2^T \\ H^T \end{bmatrix} P[A_1+E_1\Delta_1F_1 \quad A_2+E_2\Delta_2F_2 \quad H] - \begin{bmatrix} Q_1-\gamma^{-1}G^TG & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \gamma I \end{bmatrix} < 0 \quad (A23)$$

which implies $J(t, k) < 0$ for $t, k = 1, 2, \ldots$. For any integers $N_1, N_2 > 0$, from condition 29 and the assumption that all boundary conditions of 2-D FM model Σ are zeros, it results that

$$\sum_{t=0}^{N_1} \sum_{k=1}^{N_2} \Delta V(t, k) = \sum_{t=0}^{N_1} \sum_{k=1}^{N_2} (\|x(t+1, k)\|_P - \|x(t, k)\|_{Q_1}$$

$$- \|x(t+1, k-1)\|_{Q_2}) = \sum_{t=1}^{N_1} \sum_{k=1}^{N_2-1} \|x(t, k)\|_{(P-Q_1-Q_2)}$$

$$+ \sum_{k=1}^{N_2-1} \|x(N_1+1, k)\|_{(P-Q_2)} + \|x(N_1+1, N_2)\|_P \geq 0 \quad (A24)$$

followed by

$$\sum_{t=0}^{N_1} \sum_{k=1}^{N_2} (\gamma^{-1}\|x(t, k)\|_{GG^T} - \gamma\|w(t, k)\|_I) \leq \sum_{t=0}^{N_1} \sum_{k=1}^{N_2} (\Delta V(t, k)$$

$$+ \gamma^{-1}\|x(t, k)\|_{GG^T} - \gamma\|w(t, k)\|_I) = \sum_{t=0}^{N_1} \sum_{k=1}^{N_2} J(t, k) < 0 \quad (A25)$$

that is, $\|y\|_{2e} < \gamma\|w\|_{2e}$, indicating the 2-D FM system has robust H_∞ performance γ. □

A3. Proof of Theorem 3.6

Proof: It is clearly shown from the proof of Theorem 3.4 that the sufficient condition for the robust stability of closed-loop 2-D system Σ_{cl} is that there exist PDS matrices P, Q_1, and Q_2 such that the following inequalities hold:

$$\begin{bmatrix} \bar{A}_1^T + \Delta\bar{A}_1^T \\ \bar{A}_2^T + \Delta\bar{A}_2^T \end{bmatrix} P[\bar{A}_1 + \Delta\bar{A}_1 \quad \bar{A}_2 + \Delta\bar{A}_2] - \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} < 0 \quad (A26)$$

$$\alpha Q_1 + \beta Q_2 < P \quad (A27)$$

In terms of definition 32, $\{\Delta\bar{A}_i\}_{i=1,2}$ can be written as

$$\Delta\bar{A}_i = \bar{E}_i\Delta_i\bar{F}_i + \bar{E}_3\Delta_3\bar{F}_{3i}, \quad i = 1, 2 \quad (A28)$$

where

$$\bar{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \quad \bar{F}_i = [F_i \quad 0], \quad \bar{E}_3 = \begin{bmatrix} E_3 \\ 0 \end{bmatrix},$$

$$\bar{F}_{3i} = [F_3 D_{ci} C \quad F_3 C_{ci}], \quad i = 1, 2 \quad (A29)$$

By repeatedly applying Lemmas A1–A3, it is easily concluded that condition A26 is equivalent to the following condition:

$$\begin{bmatrix} -P & P\bar{A}_1 & P\bar{A}_2 & P\bar{E}_1 & P\bar{E}_2 & P\bar{E}_3 & 0 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 & 0 & F_1^T & 0 & F_{31}^T \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & F_2^T & F_{32}^T \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1^{-1} I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2^{-1} I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_3^{-1} I \end{bmatrix} < 0 \quad (A30)$$

Let

$$P \triangleq \begin{bmatrix} Y & N \\ N^T & W \end{bmatrix}, \quad P^{-1} \triangleq \begin{bmatrix} X & M \\ M^T & Z \end{bmatrix},$$

$$\Lambda_1 \triangleq \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \quad \Lambda_2 \triangleq \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \quad (A31)$$

It is easy to verify that $\Lambda_1^T P \Lambda_1 = \Lambda_1^T \Lambda_2 = \Lambda_2^T \Lambda_1$, and $I - XY = MN^T$. Now, by pre- and postmultiplying nonsingular matrix $\bar{\Lambda}^T = \mathrm{diag}\{\Lambda_1^T, \Lambda_1^T, \Lambda_1^T, I, I, I, I, I, I\}$ and $\bar{\Lambda}$, the inequality A30 is equivalent to the following condition:

$$\bar{\Lambda}^T \begin{bmatrix} -P & P\bar{A}_1 & P\bar{A}_2 & P\bar{E}_1 & P\bar{E}_2 & P\bar{E}_3 & 0 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 & 0 & F_1^T & 0 & F_{31}^T \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & F_2^T & F_{32}^T \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1^{-1} I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2^{-1} I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_3^{-1} I \end{bmatrix} \bar{\Lambda} < 0 \quad (A32)$$

that is,

$$\begin{bmatrix} -\Lambda_1^T P \Lambda_1 & \Lambda_1^T P \bar{A}_1 \Lambda_1 & \Lambda_1^T P \bar{A}_2 \Lambda_1 & \Lambda_1^T P \bar{E}_1 & \Lambda_1^T P \bar{E}_2 & \Lambda_1^T P \bar{E}_3 & 0 & 0 & 0 \\ * & -\Lambda_1^T Q_1 \Lambda_1 & 0 & 0 & 0 & 0 & \Lambda_1^T \bar{F}_1^T & 0 & \Lambda_1^T \bar{F}_{31}^T \\ * & * & -\Lambda_1^T Q_2 \Lambda_1 & 0 & 0 & 0 & 0 & \Lambda_1^T \bar{F}_2^T & \Lambda_1^T \bar{F}_{32}^T \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1^{-1} I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2^{-1} I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_3^{-1} I \end{bmatrix} < 0 \qquad \text{(A33)}$$

By setting $\{S_i = \Lambda_1^T Q_i \Lambda_1\}_{i=1,2}$, the following results can be derived from Eq. 36:

$$\Lambda_1^T P \Lambda_1 = \mathbf{\Omega}_{11}, \quad \Lambda_1^T P \bar{A}_i \Lambda_1 = \mathbf{A}_i, \quad \Lambda_1^T P \bar{E}_j = \mathbf{E}_j, \quad \Lambda_1^T \bar{F}_i^T = \mathbf{F}_i,$$
$$\Lambda_1^T \bar{F}_{3i}^T = \mathbf{F}_{3i} \quad i = 1, 2, \quad j = 1, 2, 3 \qquad \text{(A34)}$$

Therefore, Eq. A26 ⇔ Eq. A33. Also, by pre- and postmultiplying nonsingular matrix Λ_1^T and Λ_1, it is clear seen that Eq. A27 ⇔ Eq. 34, thus reaching the conclusion. □

A4. Proof of Theorem 3.7

Proof: From the proof of Theorem 3.5, it is clearly seen that the closed-loop 2-D FM system Σ_{cl} has robust H_∞ performance γ if the following conditions are satisfied:

$$\begin{bmatrix} \bar{A}_1^T + \Delta \bar{A}_1^T \\ \bar{A}_2^T + \Delta \bar{A}_2^T \\ \bar{H}^T \end{bmatrix} P \begin{bmatrix} \bar{A}_1^T + \Delta \bar{A}_1^T \\ \bar{A}_2^T + \Delta \bar{A}_2^T \\ \bar{H}^T \end{bmatrix}^T - \begin{bmatrix} Q_1 - \gamma^{-1} \bar{G}^T \bar{G} & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \gamma I \end{bmatrix} < 0 \qquad \text{(A35)}$$

$$\alpha Q_1 + \beta Q_2 < P \qquad \text{(A36)}$$

By similar use of the proof of Theorem 3.6, it can be proved that Eq. A35 ⇔ Eq. A37 and Eq. A36 ⇔ Eq. 38, which implies the conclusion. □

A5. Iterative algorithm for solving nonlinear matrix inequality 33 or 37

Without loss of generality, consider the following matrix inequality:

$$F(x, \varepsilon, \varepsilon^{-1}) < 0 \qquad \text{(A37)}$$

where $0 < x \in \mathbb{R}^{n \times n}$, $0 < \varepsilon \in \mathbb{R}$ are decision variables. Although $F(\cdot, \cdot, \cdot)$ represents a linear matrix function, Eq. A37 is a nonlinear matrix inequality arising from the nonlinear term ε^{-1}. Obviously, any algorithms used to solve this inequality can be directly extended to solve the matrix inequality 33 or 37.

Now, consider the following optimization problem with nonlinear objective function subject to LMI conditions:

$$\underset{x, \varepsilon_1, \varepsilon_2}{Minimize}\ J = Trace(\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_1) \qquad \text{(A38)}$$

$$Subject\ to\ F(x, \varepsilon_1, \varepsilon_2) < 0, \quad \begin{bmatrix} \varepsilon_1 & 1 \\ 1 & \varepsilon_2 \end{bmatrix} \geq 0 \qquad \text{(A39)}$$

Obviously, if the above minimization problem has a feasible solution and the optimal value equals 2, it can be claimed that problem A37 has a feasible solution. Although it may not be able to find the global optimal solution, the above nonlinear minimization problem is easier to solve than nonlinear matrix inequality problem A37. By using the linearization method,[27] the following iterative algorithm can be constructed. Note that $F(x, \varepsilon_1, \varepsilon_1^{-1}) < 0$ is used as a stopping criterion in the algorithm because it is difficult to obtain a feasible solution such that $J = \text{Trace}(\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_1)$ is exactly equal to 2.

Algorithm A1

Step 1. If a feasible solution $\{x_0, \varepsilon_{1,0}, \varepsilon_{2,0}\}$ satisfying LMI condition A39 can be found, then set $k = 0$, $J_0 = \varepsilon_{1,0}\varepsilon_{2,0} + \varepsilon_{2,0}\varepsilon_{1,0}$, and go to next step; otherwise, there is no feasible solution for matrix inequality A37.

Step 2. Solve the following linear minimization problem involving LMI conditions for the variables $\{x, \varepsilon_1, \varepsilon_2\}$:

$$\underset{x, \varepsilon_1, \varepsilon_2}{Minimize}\ J_k = Trace(\varepsilon_{1,k} \varepsilon_2 + \varepsilon_{2,k} \varepsilon_1) \qquad \text{(A40)}$$

$$Subject\ to\ F(x, \varepsilon_1, \varepsilon_2) < 0, \quad \begin{bmatrix} \varepsilon_1 & 1 \\ 1 & \varepsilon_2 \end{bmatrix} \geq 0$$

Set $\varepsilon_{1,k+1} = \varepsilon_1$, $\varepsilon_{2,k+1} = \varepsilon_2$, and $J_{k+1} = \varepsilon_{1,k+1}\varepsilon_{2,k+1} + \varepsilon_{2,k+1}\varepsilon_{1,k+1}$.

Step 3. If the condition $F(x, \varepsilon_1, \varepsilon_1^{-1}) < 0$ is satisfied, then $\{x, \varepsilon_1\}$ is a feasible solution of Eq. A37 and exit the iteration. If the condition $F(x, \varepsilon_1, \varepsilon_1^{-1}) < 0$ is not satisfied and $|J_k - J_{k+1}| > \sigma$, where σ is a predetermined small value, then set $k = k + 1$ and go to Step 2; otherwise, declare no feasible solution of Eq. A37.

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